Weak chaos in one-dimensional quantum transport: The $1/f^2$ law and the breakdown of the law of large numbers

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(Received 15 February 1994)

We study quantum transports in the one-dimensional Kronig-Penny model in a static electric field. S matrices as a function of the number of barriers are examined in the complex plane. They show a stagnant chaos around torus in a weak field case, while, in a strong field case, wandering from one stagnant region to another in an unpredictable way. The power spectra of transmission coefficients show a universal $1/f^2$ behavior with the exponent independent on both the strength of field and width of barriers. The Allan variance indicates a breakdown of the law of large numbers.

PACS number(s): 03.50.$-$z, 05.45.+b, 72.10.$-$d

Despite the accumulation of considerable works on quantum mechanics of chaotic systems, there exists a prevailing belief that, due to the linearity of the Schrödinger equation, the quantum system exhibits no chaos characterized by standard diagnostics of Kolmogorov-Sinai entropy or positive Lyapunov exponents. We can therefore envisage merely a quantum analogue of chaos, i.e., quantum chaology, rather than the genuine quantum chaos [1].

On the other hand, a rapid progress in fabrication of nanoscale structures has made it possible to see typical quantum-mechanical effects such as ballistic transports and tunnelings [2,3]. In particular, a growing attention has been paid to superlattices (e.g., GaAs/AlAs) with alternating sequence of potential barriers and wells. In the periodic superlattice in an applied electric field, resonant tunneling conditions are broken and the transition occurs from extended to localized "Wannier-Stark" states. While some recent studies on the quantum transport in one-dimensional structures indicate complicated transmission properties, most of them assume either one of hierarchical (e.g., Fibonacci-type) and random potentials.

In this Brief Report, we shall analyze a weak chaos in the quantum transport in a strictly regular superlattice, i.e., the Kronig-Penny model in the constant electric field. S matrices will be studied as a function of the number of barriers. Although Jauslin first considered a δ-function limit of this model [4], he showed neither the universal integer exponent of the power spectra nor striking properties of S matrices in the complex plane.

The Hamiltonian under consideration is given by

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)-\mathcal{F}x\ , \tag{1}$$

with the periodic potential: $V(x)=V_0$ for $(j-1)a\le x\le(j-1)a+d$ and $=0$ for $(j-1)a+d\le x\le ja$ with $j=1,2,\ldots$. In (1), the last term of right hand side represents the coupling with an electric field $\mathcal{F}$. For simplicity, $\mathcal{F}x$ is here replaced by a stairwise function $\mathcal{F}\hat{x}=\mathcal{F}\sum_{j=1}^{n}\theta(x-ja)$ with $\theta(x)$ for step function.

Suppose $\Psi_0=\exp(ik_0x)+S_{11}(n)\exp(-ik_0x)$ as a sum of the incident and reflected waves for $x<0$ and $\Psi_n=S_{12}(n)\exp(ik_nx)$ as the transmitted wave after the nth barrier. Considering the over-barrier tunneling throughout, the wave functions at the jth $(1\le j\le n)$ barrier and well are given by $\Psi_j=C_je^{i\kappa_jx}+D_je^{-i\kappa_jx}$ and $A_je^{ik_jx}+B_je^{-ik_jx}$, respectively, where $\kappa_j=\sqrt{\kappa_0^2+F(j-1)}$ and $k_j=\sqrt{k_0^2+Fj}$ with $F=2m\mathcal{F}/\hbar^2$ and $(\kappa_0)^2=(k_0)^2-2mV_0/\hbar^2$. Owing to the continuity and smoothness conditions at the barrier-well junction points, we have a conservative discrete map between successive set of coefficients as

$$\begin{bmatrix}A_j\\B_j\end{bmatrix}=M_j\begin{bmatrix}A_{j-1}\\B_{j-1}\end{bmatrix}, \tag{2a}$$

where the transfer matrix M_j is expressed by

$$M_j=\begin{bmatrix}e^{-ik_j(j-1)a} & 0\\ 0 & e^{ik_j(j-1)a}\end{bmatrix}\begin{bmatrix}\alpha_{1j} & \beta_{1j}\\ \beta_{1j} & \alpha_{1j}\end{bmatrix}\times\begin{bmatrix}e^{i\kappa_jd} & 0\\ 0 & e^{-i\kappa_jd}\end{bmatrix}\begin{bmatrix}\alpha_{2j} & \beta_{2j}\\ \beta_{2j} & \alpha_{2j}\end{bmatrix}\times\begin{bmatrix}e^{ik_{j-1}[(j-1)a-d]} & 0\\ 0 & e^{-ik_{j-1}[(j-1)a-d]}\end{bmatrix} \tag{2b}$$

with $\alpha_{1j}=(k_j+\kappa_j)/(2k_j)$, $\alpha_{2j}=(\kappa_j+k_{j-1})/(2\kappa_j)$, $\beta_{1j}=(k_j-\kappa_j)/(2k_j)$, and $\beta_{2j}=(\kappa_j-k_{j-1})/(2\kappa_j)$. It should be noted in the δ-function limit of walls ($d\to0$ with $V_0d=1$), the map in (2) reduces to the greatly simplified version identical to that for barrier penetrations [4]. By iterating (2) under the boundary condition $A_0=1$, $B_0=S_{11}$ and $A_n=S_{12}$, $B_n=0$, one obtains $S_{12}(n)=\det Q_n/(Q_n)_{22}$ and $S_{11}(n)=-(Q_n)_{21}/(Q_n)_{22}$ with $Q_n=\prod_{j=1}^{n}M_j$. The transmission and reflection coefficients are given by $T_n=(k_n/k_0)|S_{12}(n)|^2$ and $R_n=|S_{11}(n)|^2$, respectively. The electric conductance is simply $\sigma_n=(2e^2/h)T_n$. The validity of our computations will be justified by noting the unitarity $(T_n+R_n=1)$.

Keeping fixed both the periodicity $a\,(=1)$ and the area of each barrier $V_0d\,(=1)$, we shall present numerical results first in the limiting case $d/a \to 0$ and then in general cases $d/a \neq 0$. In the absence of an electric field, Bloch bands are formed due to the translational lattice symmetry. Transmission coefficient T_n shows periodic oscillations for energies $(\hbar k_0)^2/2m$ belonging to allowed bands. When the electric field is switched on, the following features emerge.

(I) A weak field case $F \ll F_c(=(\pi/a)^2)$: Here Bloch bands are slightly tilted and k_n value increases within a single band. The number of Zener tunnelings at the zone boundary $k_{BZ}(=\pi l/a$ with $l=1,2,\ldots,)$ is practically vanishing. T_n shows however, a nonstationary erratic behavior around the plateau value as displayed in Fig. 1(a). Reflection component of S matrix, $S_{11}(n)$, wanders in an erratic way [see Fig. 1(b)] in the complex plane. Its motion is stagnant around the torus rather than exhibiting a global chaos. This fact leads to the absence of the positive Lyapunov exponent, although the distribution function of local Lyapunov exponents will have positive components. Corresponding to this peculiar feature, the power spectra for the time sequence $\{T_n\}$ defined by $P(f)=|N^{-1}\sum_{n=1}^{N}T_n e^{-2\pi i f n/N}|^2$ exhibit $1/f^{\nu}$ law [5] with the integer exponent $\nu=2.000\pm0.001$ [see Fig. 1(c)], which is reminiscent of Brownian motions. The $1/f^{\nu}$ law with $\nu>0$ is often generated in classical dynamical systems of weakly ergodic class, e.g., in the intermittent chaos and general Hamiltonian systems, and is called simply as $1/f$ noise [5]. Despite recent active works on $1/f$ noise in classical systems [6], little attention has been given to $1/f$ noise in *quantum systems.*

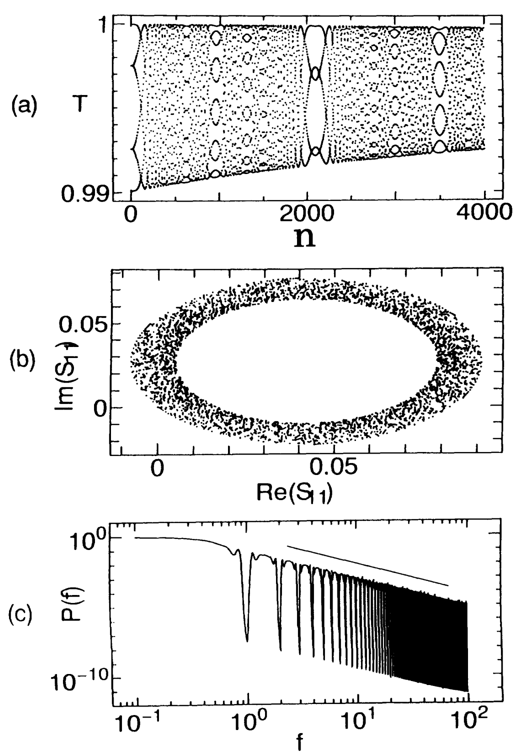

FIG. 1. Transport properties of δ-function model for $F=0.001$: (a) T_n; (b) $S_{11}(n)$; (c) power spectrum $P(f)$ of (a) in logarithmic scales, including a reference line corresponding to $P\propto f^{-2}$. ($k_0=10$ is used throughout in Figs. 1–4.)

The results in Fig. 1 are mysterious, if one should recall the lack of extrinsic randomness introduced into the system and the linearity of the Schrödinger equation [7]. A clue to resolving the puzzle will be involved in the nonautonomous feature of the map (2). With increasing the field strength, more interesting issues will come out as described below.

(II) A strong field case $(F \gg F_c)$:T_n shows a sequence of steadily-elongated plateaus which are connected by bursts. [See Fig. 2(a).] Both the direction (upwards or downwards) and the magnitude of each burst are not predictable, which is in marked contrast with the feature of the intermittent chaos where only the location of bursts is erratic. The component $S_{11}(n)$ in Fig. 2(b) wanders from one stagnant region to another in an unpredictable way, whose overall feature looks like a living animal. As in the case (I), however, the positive Lyapunov exponent is vanishing. The power spectrum of Fig. 2(a) again shows the $1/f^2$ law [Fig. 2(c)]. Locally, a picture of the tilted band structure is meaningful. Laminar oscillations in the plateaus [see the inset of Fig. 2(a)] are caused by the increase of k_n values within each band, whereas bursts are due to the Zener tunneling to adjacent bands at the zone boundaries. In fact, we recognize that the burst occurs whenever $k_n=k_{BZ}$ is satisfied.

We shall now proceed to examine the transport properties in generic systems with barriers of finite width for the cases of $d/a=0.1$ and 0.5. In this case the problem is actually the over-barrier transmission. Despite this fact, a gross feature of T_n for $F=10$ is identical to Fig. 2(a) (see Fig. 3) and corresponding power spectrum obeys the

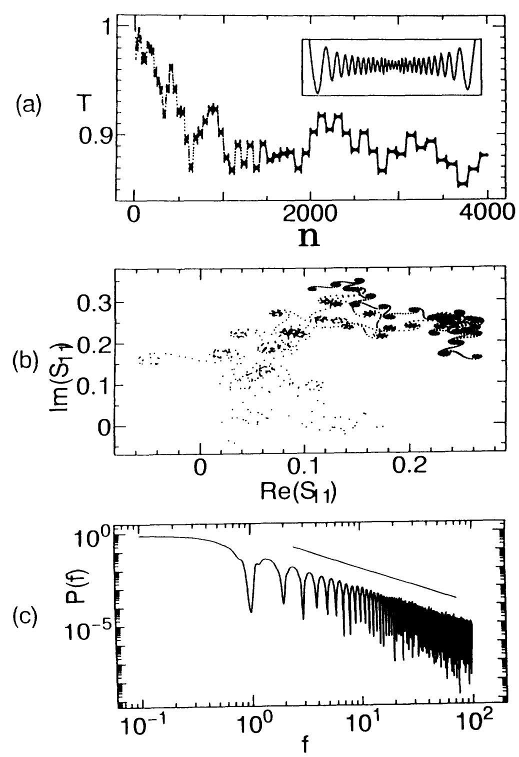

FIG. 2. Same as in Fig. 1, but for $F=10$.

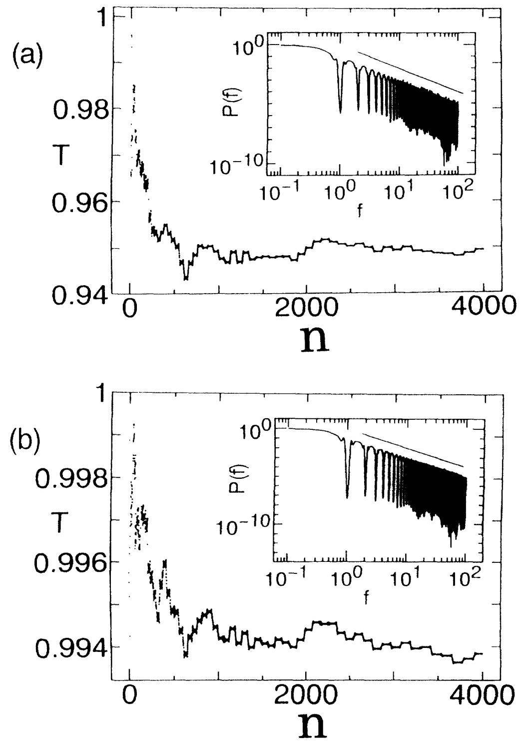

FIG. 3. Transport properties for models with barriers of finite width: (a) $d/a=0.1$; (b) $d/a=0.5$. Insets display the corresponding power spectra.

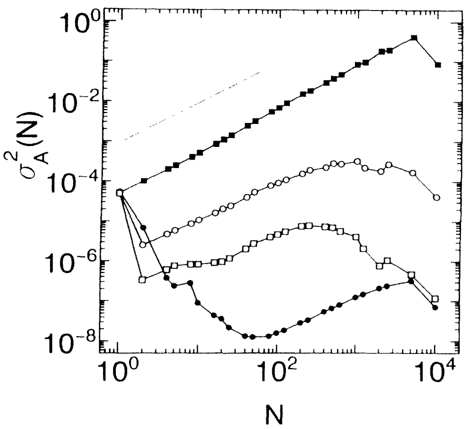

FIG. 4. Allan variance $\sigma_A^2(N)$ in logarithmic scales, including a reference line corresponding to $\sigma_A^2(N)\propto N^1$. Symbols are used for δ-function model with $F=0.001$ (●) and with $F=10$ (○) and for $d/a=0.1$ with $F=0.001$ (■) and with $F=10$ (□).

$1/f^2$ law (see insets of Fig. 3). Thus the integer exponent in the $1/f^2$ law holds universally to periodic superlattices in the electric field, irrespective of the field strength and the width of barriers.

To characterize the nonstationary behavior of T_n, we shall calculate the Allan variance [8]

$$\sigma_A^2(N)=\frac{1}{2}\left\langle\left[\frac{1}{N}\sum_{n=1}^{N}T_n-\frac{1}{N}\sum_{n=1}^{N}T_{n+N}\right]^2\right\rangle . \quad (3)$$

In the Markovian process, the variance in (3) tends to zero as N is increased, satisfying a scaling law $\sigma_A^2=N^\gamma$ with $\gamma<0$. In the $1/f^\nu$ cases, however, an additional scaling region that breaks the law of large numbers is proposed [8]: it is the fractional noise regime with $\gamma=\nu-1$. Figure 4 shows, besides the Markovian regime, there generally appears this scaling regime with the exponent $\gamma=1$ $(=2-1)$. Eventually the result provides an additional justification of the universality of $1/f^2$ law in the present system and also indicates the breakdown of the law of large numbers.

To design the experiments, we should systematically change quantum systems by increasing the length of superlattices by a step of a, against repeated injections of electrons with the fixed k_0. It should be noted that we have also analyzed the transmission coefficient as a function of k_0 under the fixed number of walls, finding again the $1/f^2$ law. The latter case would be experimentally more accessible and will be described elsewhere [9].

In conclusion, the electric conductance as a function of the number of barriers shows a nonstationary weak chaos characterized by both the $1/f^\nu$ law and anomalous Allan variance. In particular, we have found the integer exponent $\nu=2$ in marked contrast with $\nu=1.6$ estimated by Jauslin [4] and also showed the universality of this integer exponent by tuning both the width of barriers and strength of the applied field. We should emphasize many puzzling features of S matrices in the tunneling-induced weak chaos in quantum systems.

The authors greatly acknowledge the latest fruitful correspondence with H. R. Jauslin now at Universite de Bourgogne and useful discussions with Y. Aizawa and T. Ogawa.

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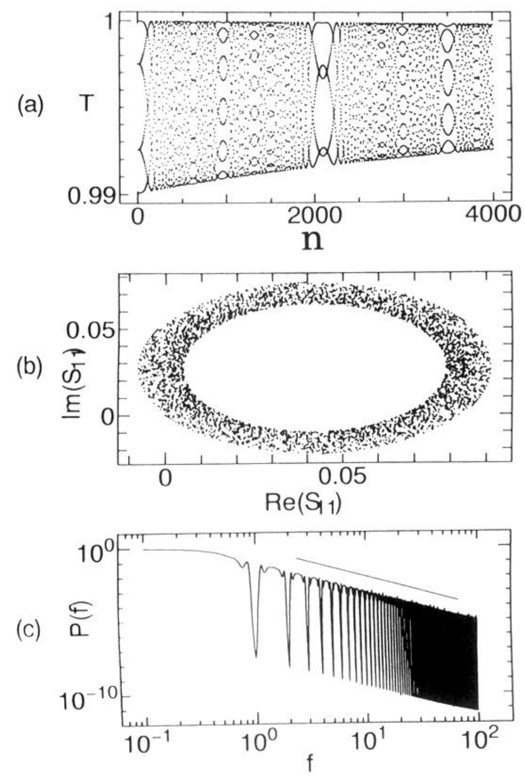

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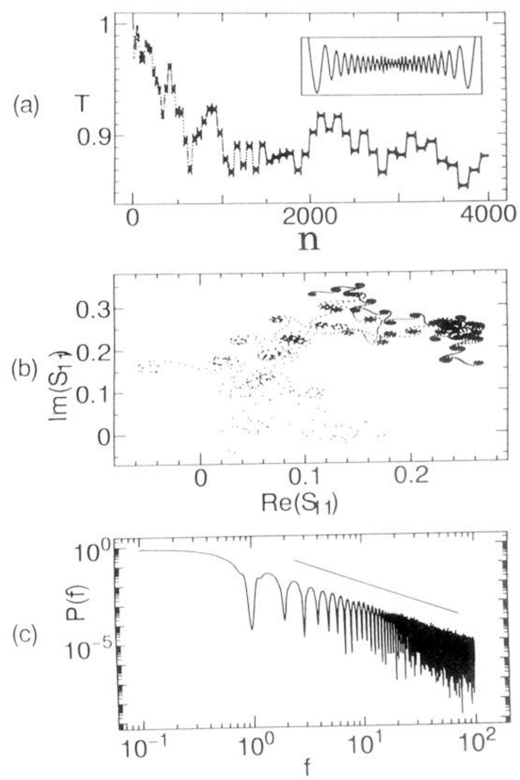

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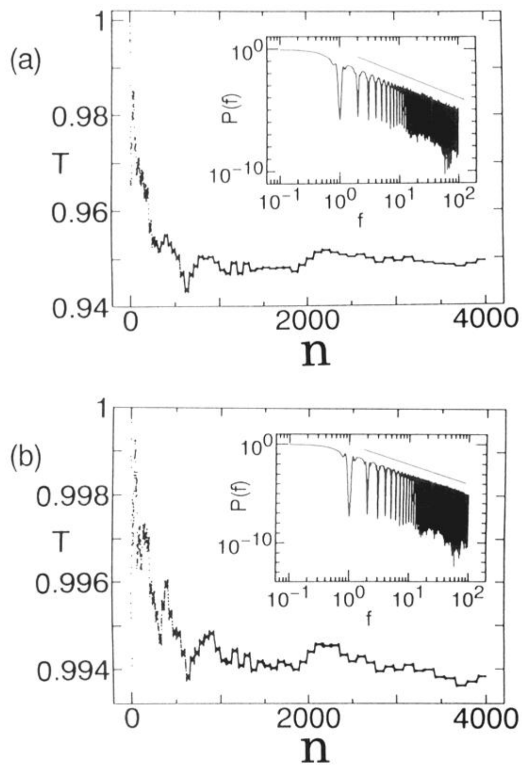

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